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Polar cographs

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Abstract

Polar graphs are a natural extension of some classes of graphs like bipartite graphs, split graphs and complements of bipartite graphs. A graph is (s, k)-polar if there exists a partition A, B of its vertex set such that A induces a complete s-partite graph (i.e., a collection of at most s disjoint stable sets with complete links between all sets) and B a disjoint union of at most k cliques (i.e., the complement of a complete k-partite graph).

Recognizing a polar graph is known to be *NP*-complete. These graphs have not been extensively studied and no good characterization is known. Here we consider the class of polar graphs which are also cographs (graphs without induced path on four vertices). We provide a characterization in terms of forbidden subgraphs. Besides, we give an algorithm in time $O(n)$ for finding a largest induced polar subgraph in cographs; this also serves as a polar cograph recognition algorithm. We examine also the monopolar cographs which are the (s, k)-polar cographs where $\min(s, k) \leqslant 1$. A characterization of these graphs by forbidden subgraphs is given. Some open questions related to polarity are discussed.

Keywords: Polar graphs; Cographs; Split graphs; Threshold graphs

1. Introduction

Polar graphs are a natural extension of some classes of graphs which include bipartite graphs, split graphs (i.e., graphs whose vertex set can be partitioned into a clique and a stable set) and complements of bipartite graphs.

Following [2], a graph $G = (V, E)$ is called *polar* if its vertex set V can be partitioned into (A, B) (A or B may possibly be empty) such that A induces a complete multipartite graph (it is a join of stable sets) and B a (disjoint) union of cliques (i.e., the complement of a join of stable sets).

We shall say that G is (s, k)-*polar* if there exists a partition (A, B) where A induces a join of at most s stable sets and B a union of at most k cliques. Thus polar graphs are just the (∞, ∞)-polar graphs. Notice that not every graph is polar: the graphs N_1 and N_2 in Fig. 1 are not polar as can be checked, but if any vertex is removed, the remaining graph is polar. Observe also that the complement $\overline{G}$ of an (s, k)-polar graph is a (k, s)-polar graph. Notice that $(1, 1)$-polar graphs are just split graphs.

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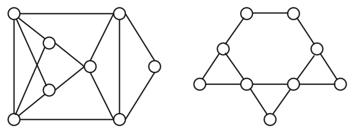

Fig. 1. Some minimal non-polar graphs.

In [2] it was shown that recognizing whether an arbitrary graph is polar is *NP*-complete. These graphs have not been extensively studied. Some polynomial time recognition problems are discussed in [2] for the case where the largest size of the stable sets and of the cliques in the partition (A, B) are bounded. Besides this, [5] gives a general framework for partitioning the vertex set of graphs with requirements on the links between the subsets of the partitions.

The question that arises is to find subclasses of polar graphs which can be recognized in polynomial time and for which nice characterizations can be found. It turns out that for cographs one can derive such results; this is a first step which could be followed by various extensions.

We recall that *cographs* are the graphs without induced P_4 (path on four vertices). It follows from this definition that G is a cograph if and only if its complement $\overline{G}$ is a cograph. It is well known [3] that for a cograph G, either G or $\overline{G}$ is disconnected.

We study polar cographs and give a characterization by forbidden subgraphs in Section 2 as well as a polynomial time recognition algorithm in Section 4.

We will also examine a subclass of polar graphs called *monopolar* graphs; these are the (s, k)-polar graphs where $\min(s, k) \leqslant 1$. In other words for such graphs, a partition (A, B) exists with at most one stable set in $G[A]$ or at most one clique in $G[B]$. A characterization of monopolar cographs by forbidden subgraphs will be derived in Section 3.

In addition, some remarks on the recognition of (s, k)-polar graphs will be provided in Section 5 as well as some open questions related to other classes of polar graphs.

In what follows, we denote by P_l, C_l and K_l respectively a path, a chordless cycle and a clique on l vertices. Given two graphs G_1, G_2, $G_1 \oplus G_2$ denotes their join (with complete links) and $G_1 \cup G_2$ their disjoint union. Let x, y be two vertices, then xy and $\overline{xy}$ mean respectively that x is linked to y and x is not linked to y.

We will also need the notion of *threshold graphs* which are split graphs (i.e, (1, 1)-polar graphs), where for any two vertices v, w in the stable set S, the sets of neighbors satisfy $N(v) \supseteq N(w)$ or $N(w) \supseteq N(v)$. In other words, the vertices of the stable set can be linearly ordered by domination (i.e., inclusion of neighborhoods). A graph G is a threshold graph if and only if it does not contain $2K_2$, C_4 or P_4 as induced subgraphs. Properties of threshold graphs are studied in [6]. Notice that threshold graphs are precisely the split cographs.

It will be convenient to call *complete* (s, k)-polar an (s, k)-polar graph with partition (A, B), which is the join of A and B (i.e., with complete links between A and B).

Throughout the paper, a set of vertices and the graph induced by such a set will occasionally be identified for the sake of notational simplicity; this should be clear from the context. All connected components of graphs will simply be called components whenever no confusion arises. For graph theoretical terms not defined here, the reader is referred to [1].

2. Characterization of polar cographs by forbidden subgraphs

In this section, we provide a forbidden subgraph characterization of polar cographs. Since there is a finite family of forbidden subgraphs, there is an obvious polynomial time recognition algorithm. We will however describe in Section 4 a recognition algorithm with better time complexity.

Theorem 1. *For a cograph G, the following statements are equivalent*:

(a) *G is polar*.
(b) *Neither G nor $\overline{G}$ contains any of the graphs $H_1, \ldots, H_4$ of Fig. 2 as induced subgraphs.*

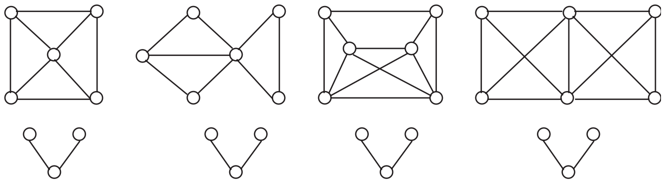

Fig. 2. Forbidden subgraphs for polar cographs.

Proof. Notice first that the subgraphs H_i can be viewed as follows:

(1) $H_1 = P_3 \cup (\overline{K}_2 \oplus P_3)$
(1′) $H_1 = P_3 \cup (K_1 \oplus C_4)$
(2) $H_2 = P_3 \cup (K_1 \oplus (P_3 \cup K_2))$
(3) $H_3 = P_3 \cup (\overline{P}_3 \oplus \overline{P}_3)$
(4) $H_4 = P_3 \cup (K_2 \oplus 2K_2)$

(a) $\Rightarrow$ (b) Since every induced subgraph of a polar graph is polar and the complement of a polar graph is polar, it is enough to show that $H_1, \ldots, H_4$ are non-polar. Suppose $H_1, \ldots, H_4$ are polar. Clearly, none of $H_1, \ldots, H_4$ is $(1, k)$-polar for any k. Since a complete join of stable sets is a connected subgraph, it follows that in each of the four graphs, all the components except for one must be a clique. Clearly, it is not the case for any of the graphs. Hence they are non-polar.

(b) $\Rightarrow$ (a) Suppose G is non-polar. Assume without loss of generality that G is minimal non-polar. Assume also without loss of generality that the cograph G is disconnected (otherwise take its complement). Let (A, B) be a partition of its vertex set into non-empty sets without edges between A and B. By the minimality of G, both $G[A]$ and $G[B]$ are polar. If $G[A]$ contains no induced P_3, then it is a disjoint union of cliques and hence G is polar. So we may assume that both A and B contain three vertices inducing a P_3. If both $G[A]$ and $G[B]$ have polar partitions with single stable sets S_A and S_B respectively, then G has a polar partition with single stable set $S_A \cup S_B$. So assume that $G[A]$ has at least two stable sets in every polar partition. Let $A' \subseteq A$ induce the connected component of $G[A]$ containing the join of stable sets in a possible polar partition of A. Note that $G[A']$ may also contain some cliques of this polar partition, and that $A \backslash A'$ induces a disjoint union of cliques. Since G is a cograph, A' is partitioned into (C, D) with complete links between C and D. We consider two cases.

Case 1: An induced P_3 in A is completely contained in D.

If C contains a $\overline{K}_2$, then the $\overline{K}_2$ along with the P_3 in D and the P_3 in B induce an $H_1 = P_3 \cup (\overline{K}_2 \oplus P_3)$, a contradiction. So C must induce a clique. If C contains an edge, then D is $2K_2$-free for otherwise G is isomorphic to H_4 (see item 4). D is also C_4-free (else G is isomorphic to $H_1 = P_3 \cup (K_1 \oplus C_4)$) and P_4-free. Thus D induces a threshold graph. It follows that $G[A]$ has polar partition with at most one stable set and possibly several cliques (the clique K formed by the one in D which is completely linked to the clique induced by C, and the cliques in $A \backslash A'$ which are not linked to K since A' is a connected component of A), a contradiction. It follows that C must consists of a single vertex u; so $C = \{u\}$.

Let S be a maximal stable set in D containing both ends v, w of P_3 in D. Let c be the center of the P_3. Let $N(c)$ be the set of the neighbors of c in D.

Claim 2. *For every $a \in D \backslash S$, ac, i.e., $D \backslash S \subseteq N(c)$.*

Proof. S being a maximal stable set, a has a neighbor y in S. If $a \notin N(c)$, then we consider several cases. If ($y = v$ and aw) or ($y = v$ and $\overline{aw}$) then D induces a C_4 or a P_4, respectively. Otherwise, we have necessarily $\overline{aw}$ and $\overline{av}$. Then if yc, D induces a P_4; if $\overline{yc}$, D induces $P_3 \cup K_2$. All these cases imply contradictions since G is P_4-free and is not isomorphic to $H_1 = P_3 \cup (K_1 \oplus C_4)$ or $H_2 = P_3 \cup (K_1 \oplus (P_3 \cup K_2))$. □

Claim 3. *For every $a \in D \backslash S$, there exists $x \in N(c) \cap S$ such that ax.*

Proof. By Claim 2, we have ac for every $a \in D \backslash S$. Then the claim follows from the fact that G is P_4-free. □

Claim 4. *$N(c) \cap S$ is linearly ordered by domination in $N(c) \backslash S$, i.e., there are no two vertices $x, y \in N(c) \cap S$ such that for some $a, b \in N(c) \backslash S$ xa, $\overline{xb}$, $\overline{ya}$ and yb.*

Proof. Since $a, b \in N(c) \backslash S$, ac and bc. If $\overline{ab}$ then a, b, c, x, y and u along with P_3 in B induce $H_4 = P_3 \cup (K_2 \oplus 2K_2)$. If ab, then G contains a P_4. □

Claim 5. *There exists $d \in N(c) \cap S$ such that da for all $a \in N(c) \backslash S$.*

Proof. Follows from Claims 3 and 4. □

Claim 6. *For any $x \in S \backslash \{d\}$ and for any $a, b \in N(c) \backslash S$, if $\overline{ab}$ then $\overline{xa}$ and $\overline{xb}$.*

Proof. Since da, db it follows that if xa and/or xb, then D contains a C_4 or P_4, a contradiction because C_4 with u and P_3 induces H_1 (see item $1'$). □

Claim 7. *$N(c) \backslash S$ is $2K_2$-free.*

Proof. Any $2K_2$ in $N(c) \backslash S$, along with cd, and a P_3 from B would induce H_4 (see item 4), a contradiction. □

Claim 8. *$G[A]$ has a polar partition with a single stable set.*

Proof. By Claim 7, $(N(c) \backslash S)$ is $2K_2$-free. Also D is C_4-free otherwise we have H_1. Hence $(N(c) \backslash S)$ induces a threshold graph. Let (S', K) be a polar partition of $(N(c) \backslash S) \cup \{u\}$ with S' the single stable set and K the single clique. Then $(S' \cup S \backslash \{d\}, K \cup \{d, c\})$ is a polar partition of $G[A']$ with a single stable set, by Claims 5 and 6. Therefore, $G[A]$ also has a polar partition with a single stable set. □

It follows that Case 1 is impossible.

Case 2: Every P_3 of A intersects both C and D.

Since both C and D are P_3-free, each one induces a disjoint union of cliques. We can assume without loss of generality that C consists of either a single clique or a single stable set, for otherwise, i.e., if both C and D are neither a single clique nor a single stable set, G is isomorphic to $H_3 = P_3 \cup (\overline{P}_3 \oplus \overline{P}_3)$. If C consists of a single stable set, then $G[A]$ has a polar partition with one stable set. If C consists of a clique of size 2, then D has at most one clique of size 2 (else G is isomorphic to H_4 (see item 4)). It follows that the rest of D forms a single stable set and $G[A]$ has a polar partition with a single stable set and possibly several cliques. Thus this case is also impossible.

It follows that G must be polar. □

3. Characterization of monopolar cographs by forbidden subgraphs

As in Section 2, we simply give here a characterization based on forbidden subgraphs; a more involved recognition algorithm will be given in Section 4.

Theorem 9. *For a cograph G, the following are equivalent*:

(a) *G is monopolar.*
(b) *Neither G nor $\overline{G}$ contains any of the graphs $G_1, \ldots, G_9$ of Fig. 3 as an induced subgraph.*
(c) *G or $\overline{G}$ is a disjoint union of threshold graphs and complete $(1, \infty)$-polar graphs.*

Proof. We first notice that the graphs G_i can be viewed as follows where P and Q are as in Fig. 4:

(1) $G_1 = 2K_1 \cup (K_1 \oplus C_4)$
($1'$) $G_1 = (2K_1 \oplus P_3) \cup 2K_1$

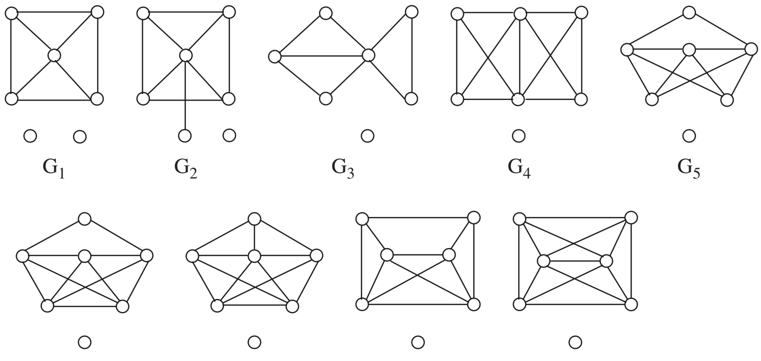

Fig. 3. Forbidden subgraphs for monopolar cographs.

(2) $G_2 = ((C_4 \cup K_1) \oplus K_1) \cup K_1$
(3) $G_3 = (Q \oplus K_1) \cup K_1$
(4) $G_4 = (2K_2 \oplus K_2) \cup K_1$
(4′) $G_4 = (P \oplus K_1) \cup K_1$
(5) $G_5 = ((P_3 \cup K_1) \oplus 2K_1) \cup K_1$
(6) $G_6 = ((K_1 \cup K_3) \oplus 2K_1) \cup K_1$
(7) $G_7 = (P_3 \oplus (K_2 \cup K_1)) \cup K_1$
(7′) $G_7 = (K_1 \oplus H) \cup K_1$ where $H=$ is a C_4 with an additional vertex linked to three vertices of the C_4
(7″) $G_7 = ((K_1 \oplus (K_1 \cup K_2)) \oplus 2K_1) \cup K_1$
(8) $G_8 = ((K_2 \cup K_1) \oplus (K_2 \cup K_1)) \cup K_1$
(9) $G_9 = ((C_4 \oplus K_1) \oplus K_1) \cup K_1$
(9′) $G_9 = ((K_4 \backslash e) \oplus 2K_1) \cup K_1$
(9″) $G_9 = (P_3 \oplus P_3) \cup K_1)$

(a) $\Rightarrow$ (b) Since the complement of a monopolar graph and every induced subgraph of a monopolar graph are also monopolar, it is enough to show that $G_1, \ldots, G_9$ are not monopolar. Since the non-trivial component in each one of these graphs is not a clique, it must contain the join of stable sets in any polar partition. It is routine to verify that any polar partition of these graphs must be the join of at least two stable sets and the union of at least two cliques. Hence they are not monopolar.

(c) $\Rightarrow$ (a) First, note that a threshold graph has a polar partition into a single stable set and a single clique, and disjoint union of stable sets is a single stable set. It follows that if G is a disjoint union of threshold graphs and complete $(1, \infty)$-polar graphs, then G is monopolar with a single stable set and a disjoint union of cliques.

(b) $\Rightarrow$ (c) Since G is a cograph, assume without loss of generality that G is disconnected. It is enough to show that each non-trivial component of G is either a threshold graph or a complete $(1, \infty)$-polar graph. Let G' be any non-trivial component of G. Further assume that G' is the join of A, B (i.e., $G' = G[A] \oplus G[B]$). The non-empty sets A, B exist since G' is a connected cograph with at least two vertices. We consider several cases. Recall that G has at least two components.

Case 1: A induces a C_4 with vertices a, b, c, d and edges ab, bc, cd and ad.

B must be a stable set, for otherwise G contains $G_9 = (C_4 \oplus K_2) \cup K_1$. Let x be any other vertex of A. Then

(i) x must be linked to at least one vertex of the C_4, for otherwise G contains a $G_2 = ((C_4 \cup K_1) \oplus K_1) \cup K_1$,
(ii) x may not be linked to exactly three vertices of the C_4, for otherwise G contains a G_7 (see item 7′),
(iii) x may not be linked to all four vertices of the C_4, for otherwise G contains $G_9 = ((C_4 \oplus K_1) \oplus K_1) \cup K_1$ and
(iv) x may not miss two linked vertices of the C_4 for otherwise G' contains P_4.

It follows that each vertex of A other than a, b, c, d is linked either to a and c or else is linked to b and d. Let $N(a)$ be the set of the neighbors of a in A and $N(b)$ be the set of the neighbors of b in A. Clearly $N(a)$ and $N(b)$ form

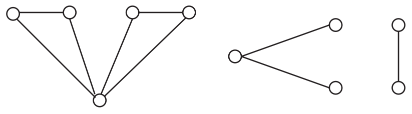

Fig. 4. Case 2 of Theorem 9.

stable sets by (i)–(iv) and because if two vertices x and x' in $N(a)\backslash C_4$ (or in $N(b)\backslash C_4$) are linked then G contains a $G_7 = (K_1 \oplus H) \cup K_1$ (see item 7′). Also, $N(a)$ and $N(b)$ are completely joined, to avoid induced P_4. Thus G' is the join of three stable sets $N(a)$, $N(b)$ and B. Now G may contain at most one other component which must be a clique, for otherwise G contains $G_1 = 2K_1 \cup (K_1 \oplus C_4)$. It follows that the complement $\overline{G}$ is a complete $(1, 3)$-polar graph in this case, as required.

Case 2: A induces a $2K_2$.

B must form a stable set, for otherwise G contains a $G_4 = (2K_2 \oplus K_2) \cup K_1$. If A contains an induced P_3, then to avoid a P_4, it must contain a P or a Q of Fig. 4 as an induced subgraph. If A contains a P, then G contains a $G_4 = (P \oplus K_1) \cup K_1$ and if A contains a Q, then G contains a $G_3 = (Q \oplus K_1) \cup K_1$. It follows that A is P_3-free and hence induces a disjoint union of cliques. Hence $G' = G[A] \oplus G[B]$ is a complete $(1, \infty)$-polar graph as required.

We may now assume by symmetry, that both A and B do not induce C_4, $2K_2$ and P_4 and hence form threshold graphs.

Case 3: A induces a threshold graph containing a triangle.

(i) If A induces a $K_1 \oplus P_3$, then B must be a clique or else G contains a $G_9 = ((K_1 \oplus P_3) \oplus 2K_1) \cup K_1$. Since a threshold graph joined to a clique is a threshold graph, G' is a threshold graph in this case, as required.
(ii) If A contains a vertex linked to exactly one vertex of the triangle, then B too must be a clique or else G contains a G_7 (see item 7″). Hence G' is a threshold graph as required. It follows that A induces a clique and isolated vertices.
(iii) If A forms a clique and at least one isolated vertex, then B induces no $2K_1$, or else G contains a $G_6 = ((K_1 \cup K_3) \oplus 2K_1) \cup K_1$. Thus B is a clique and hence G' is a threshold graph as required.
(iv) If A forms a single clique, then B being a threshold graph, G' too is a threshold graph as required.

Case 4: Both A and B induce threshold graphs without triangles.

(i) If A contains an induced $P_3 \cup K_1$, then B must be a clique, for otherwise G contains $G_5 = ((P_3 \cup K_1) \oplus 2K_1) \cup K_1$. Hence G' is a threshold graph, as required. So we may assume that A is either $K_{1,n}$ for some $n > 1$, or P_3-free.
(ii) If A induces a $K_{1,n}$ with $n > 1$, then B may not contain an induced P_3 (to avoid $G_9 = (P_3 \oplus P_3) \cup K_1$) and may not contain an induced $K_2 \cup K_1$ (to avoid $G_7 = (P_3 \oplus (K_2 \cup K_1)) \cup K_1$). Thus B is a single clique or a single stable set. If B is a clique, then G' is a threshold graph as required and if B is a stable set with at least two vertices, then G may contain only one other component which is a clique, or else G contains $G_1 = (2K_1 \oplus P_3) \cup 2K_1$. Hence $\overline{G}$ is a complete $(1, 3)$-polar graph as required.
(iii) Hence, by symmetry, we may assume that both A and B are P_3-free. On the other hand, both A and B may not contain $K_2 \cup K_1$ for otherwise G contains $G_8 = ((K_2 \cup K_1) \oplus (K_2 \cup K_1)) \cup K_1$. Thus, one of A and B, say B is a clique or a stable set. If B is a clique then since A is a threshold graph, G' is also a threshold graph as required. If B is a stable set, then G' is a complete $(1, \infty)$-polar graph since A induces a disjoint union of cliques.

Thus in all cases, either the complement $\overline{G}$ is a complete $(1, 3)$-polar graph or G is a disjoint union of threshold graphs and complete $(1, \infty)$-polar graphs. $\square$

4. Largest polar subgraph in cographs

Recall that for a cograph G, either G or $\overline{G}$ is disconnected. Subsequently, a tree can be constructed with cograph G as the root. Children of each vertex represent the components of either the graph at the parent vertex (in which case the parent vertex is a *0-vertex*), or the complement of the graph at the parent vertex (in which case the parent is a *1-vertex*). This tree is known as the *cotree* and can be constructed in linear time [3].

In what follows, we assume that the cotree representation of the cograph is given. If x denotes a vertex in the cotree, then $c_1x, c_2x, \ldots$ are the children of x. Each vertex x has a type $t(x)$ which is 0 or 1 and x will also represent the subgraph associated with vertex x of the cotree. This should be clear from the context.

In this section, we describe how to find an induced polar subgraph of maximum size in a cograph using its cotree representation. Given a cograph G let us denote by $MC(G)$ a maximum clique in G, by $MS(G)$ a maximum stable set in G, by $MT(G)$ a maximum threshold graph in G, by $MUC(G)$ a maximum (size) union of cliques in G, by $MJS(G)$ a maximum (size) join of stable sets in G, by $MMPS(G)$ a maximum $(1, k)$-polar subgraph for some k (maximum monopolar subgraph with one stable set) in G, by $MMPC(G)$ a maximum $(s, 1)$-polar subgraph for some s (maximum monopolar subgraph with one clique) in G and finally by $MP(G)$ a maximum polar subgraph in G. $n(MP(G))$ denotes the size of $MP(G)$ and the sizes of all other maximum subgraphs are denoted in a similar way. All maximum subgraphs mentioned below are represented by a pair (A, B) as described in the introduction.

First of all, note that given a cotree, $MS(x)$ and $MC(x)$ can be found in linear time for any x in the cotree [3]. Also, it has been shown in [4] that a maximum threshold subgraph in cographs is obtained by the union of any maximum stable set and any maximum clique since every pair of maximum stable set and maximum clique intersects. Therefore, for any vertex x of the cotree, $MT(x)$ is the subgraph induced by the vertices of $MC(x)$ and $MS(x)$, and $n(MT(x)) = n(MC(x)) + n(MS(x)) - 1$. In what follows, we assume for the sake of simplicity that $MC(x)$, $MS(x)$ and $MT(x)$ are known for any vertex x of the cotree.

A 0-vertex of the cotree represents a disconnected subgraph with components $c_1x, c_2x, \ldots,$ and a 1-vertex of the cotree represents a connected subgraph. The proofs of the following lemmas assume that all the relevant parameters are computed for the children of a 0-vertex recursively.

Lemma 10. *Given a cotree, $MUC(x)$ and $MJS(x)$ can be computed for any 0-vertex x in time linear in the number of children of x.*

Proof. Clearly, we have $MUC(x) = \cup_i MUC(c_ix)$ since cliques of different children are not linked at all. On the other hand, $MJS(x)$ is the set realizing the maximum of $[\max_i n(MJS(c_ix)); \sum_i n(MS(c_ix))]$; in fact if at least two children contribute then no more than one stable set can be taken from each child since the children of x are not linked at all. □

Lemma 11. *Given a cotree, $MMPS(x)$ and $MMPC(x)$ can be computed for any 0-vertex x in time $\mathrm{O}(p)$ where p is the number of children of x.*

Proof. Obviously, we have $MMPS(x) = \cup_i MMPS(c_ix)$ since the union of one stable set from each child yields one stable set and cliques of different children remain disjoint. For $MMPC(x)$, it is the subgraph realizing the maximum of $[n(MT(x)); \max_i n(MMPC(c_ix)); \max_{i \neq j}(n(MJS(c_ix)) + n(MC(c_jx)))]$. In fact, a maximum $(s, 1)$-polar subgraph at a 0-vertex is either a threshold graph, or the largest maximum $(s, 1)$-polar subgraph among the children, or the largest union among the children, of a maximum join of stable sets in one child and a maximum clique in another child (if both are coming from the same child then it amounts to be a maximum $(s, 1)$-polar subgraph of the child under consideration).

The time complexity is linear since the three terms can be computed in time $\mathrm{O}(p)$ including $Z = \max_{i \neq j}(n(MJS(c_ix)) + n(MC(c_jx)))$. Let $X = \max_i n(MJS(c_ix))$, with $h =$ argmax X; let $Y = \max_j n(MC(c_jx))$, with $k =$ argmax Y. If $h \neq k$, then $Z = X + Y$. If $h = k$, then $Z = \max[X + \max_{j \neq k} n(MC(c_jx)); Y + \max_{i \neq h} n(MJS(c_ix))]$. □

Lemma 12. *Given a cotree, $MP(x)$ can be computed for any 0-vertex x in time $\mathrm{O}(p)$ where p is the number of children of x.*

Proof. A maximum polar subgraph is obtained by either taking the union of a maximum polar graph (containing at least two stable sets) in one child and maximum union of cliques in other children, or taking the union of maximum $(1, k)$-polar subgraph from each child. It follows that $MP(x)$ is the subgraph realizing the maximum of $[\max_i (n(MP(c_ix)) + \sum_{j \neq i} n(MUC(c_jx))); \sum_i n(MMPS(c_ix))]$.

The time complexity of computing the second term is linear by Lemma 11. The first term, say Z can also be computed in linear time in the following way: let $C = \sum_j n(MUC(c_j x))$ and $H_i = C - n(MUC(c_i x))$. Then $Z = \max_i(n(MP(c_i x)) + H_i)$. □

Theorem 13. *For any cograph G given by its cotree, $MP(G)$ can be computed in time $O(n)$.*

Proof. One may think of an algorithm searching the cotree from the leaves to the root and computing for each vertex of the cotree a maximum polar subgraph; the one computed at the root provides $MP(G)$. By Lemma 12, one can compute a maximum polar subgraph at a 0-vertex. On the other hand, at a 1-vertex x, we know that the complement of the subgraph remaining under this vertex is a disconnected graph which can be represented by a cotree with a root of type 0. Then applying Lemma 12 and taking the complement of the resulting subgraph (thus stable sets and cliques interchange roles) gives a maximum polar subgraph at x.

The initialization of this algorithm is done by the following assignments: for a vertex x which is a leaf representing the vertex v, $MUC(x)$ and $MMPC(x)$ are of the form (A, B) where $A = \emptyset$, $B = \{v\}$, $MJS(x)$ and $MMPS(x)$ are of the form (A, B) where $A = \{v\}$, $B = \emptyset$ and $MP(x)$ is of one of these forms.

The complexity is $O(n)$ since Lemmas 11 and 12 are applied for all vertices of the cotree. □

We remark that in a cograph G with weighted vertices, a maximum weighted polar subgraph can be found in exactly the same way as previously; it suffices to replace in all lemmas the size of a subgraph by its weight which is the sum of the weights of the vertices in the subgraph.

Indeed, Theorem 13 implies a linear time recognition algorithm for polar cographs (given by their cotree); given a cograph $G = (V, E)$ where $|V| = n$, G is polar if and only if $n(MP(G)) = n$.

In a similar way, applying Lemma 11 to every vertex of the cotree yields a linear time recognition algorithm for monopolar cographs.

5. Final remarks

Let us mention a general remark on the recognition of polar graphs. Although it is *NP*-complete to recognize polar graphs in general, it becomes polynomially solvable if the numbers of stable sets and cliques in a polar partition are fixed. In [5], $\mathcal{S}$ and $\mathcal{D}$ are defined as two classes of graphs, called arbitrarily as classes of *sparse* and *dense* graphs respectively, satisfying the following conditions: both $\mathcal{S}$ and $\mathcal{D}$ are hereditary classes and there exists a constant c such that the intersection $S \cap D$ has at most c vertices for any $S \in \mathcal{S}$ and $D \in \mathcal{D}$. A *sparse–dense partition* of a graph G with respect to the classes $\mathcal{S}$ and $\mathcal{D}$, is a partition of the vertex set of G into two parts where one induces a sparse graph and the other one induces a dense graph.

Theorem 14 (*Feder et al.* [5]). *All sparse–dense partitions of a graph can be found in time $O(n^{2c+2}T(n))$ where $T(n)$ is the time for recognizing sparse and dense graphs.*

Corollary 15. *For any graph G and for fixed s, k, it can be recognized in polynomial time whether G admits a (s, k)-polar partition.*

Proof. First, note that a join of s stable sets is a sparse graph and that a union of k cliques is a dense graph. Then, one can observe that for fixed s and k, there can be at most $c = \min(s, k)$ vertices in the intersection of a $(s, 0)$-polar graph and a $(0, k)$-polar graph. Furthermore, $(s, 0)$-polar and $(0, k)$-polar graphs can be recognized in polynomial time; G is $(0, k)$-polar if and only if G does not contain an induced P_3, and it has k connected components; G is $(s, 0)$-polar if and only if $\overline{G}$ is $(0, s)$-polar. □

Note that the complexity of recognizing (s, k)-polar graphs is no more polynomial if s and k are not fixed.

We have provided algorithms and characterizations related to polar cographs. There are many questions that still remain to be answered. Among those a characterization of $(2, 2)$-polar cographs by forbidden subgraphs would be a natural continuation. Also one should explore more general subclasses of perfect graphs to characterize their polarity. Further research could focus on permutation graphs or line graphs of bipartite graphs.

An interesting related problem would be to point out a class of graphs (which is clearly not the cographs) for which computing a maximum polar subgraph is *NP*-hard while the polar recognition problem is polynomially solvable.

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